Analysis of Stress Distribution in the Sapling Tree Trunk^1

Andrew T. Leiser and John D. Kemper\textsuperscript{2}

\textit{University of California, Davis}

\textbf{Abstract.} The stress distribution in a sapling tree trunk (thin tapered beam under high degree of deflection) was determined mathematically by a computer and subsequently verified by actual trunk samples. This analysis showed that stress was minimal and was most uniformly distributed when the dimensionless variable, taper parameter \((kL)\), was approximately \(-0.60\) (where no taper = 0.00 and maximum taper = \(-1.00\)). Further experiments demonstrated that the young sapling tree grown without stakes or pruning developed such that a tree loaded at a single point corresponding to taper parameter \(-0.60\) bent in a similar manner to that tree bent by wind loading. The implications relative to cultural practices in the nursery are discussed.

The sapling tree growing naturally in temperate climates is usually tapered uniformly from base to tip with branches to the ground and it stands erect without benefit of a stake. Current nursery pruning and staking practices may alter this configuration drastically and produce trees incapable of standing upright without stakes. In extreme cases the normal tapered trunk is altered to one with no taper or to one with reverse taper (i.e., one with larger caliper at 4 or 5 ft than at the ground). The cultural practices causing this alteration have been identified and partially defined (7, 8).

Trees with little or no taper require staking for several years after planting in the landscape. This is costly and frequently results in severe damage to bark on limbs and trunk and even to deformed trunks (9). Present staking practices vary greatly and usually reflect a subjective approach to the problem.

Accepted grades and standards for nursery trees (3) specify relationships between height and caliper but ignore the parameter of taper.

A previous paper (9) pointed out some problems arising in the staked tree and presented an analysis of the stress in the trunk near the top of the stake for sapling trees as the staking height varied from ground level to the top of the tree. However, the stress distribution in the unstaked tree was not studied. We found no analysis of stress distribution in thin tapered beams at large deflections in the literature.

This paper presents an analysis of the stress in sapling trees of several degrees of taper, relates this analysis to wind loading of young trees and discusses the application of this analysis to the practical problems of: (a) selecting among cultural practices which affect trunk development, (b) writing more meaningful tree specifications, (c) choosing best staking methods, and (d) developing pruning practices for training the young tree.

\textbf{Materials and Methods}

The mathematical analysis was done by computer following which young sapling tree trunks of 4 species were selected for verification of the computer solution. The species used were container- and field-grown \textit{Ceratonia siliqua} L., carob, container grown \textit{Ginkgo biloba} L., ginkgo, field grown \textit{Liquidambar styraciflua} L., liquidambar or sweetgum, and field grown \textit{Betula verrucosa} Ehrh., European white birch. Side branches and different age of wood along the stem result in stem heterogeneity. However, preliminary studies comparing machined plexiglass specimens and carefully selected saplings indicated close agreement of the bending curves between the two. Care was used in selecting wood samples to avoid large lateral branches and immature wood. By selecting from suitably large populations, 6-8 ft specimens were obtained excluding the immature tip. These provided samples with the parameters listed in Table 1. All specimens of a species were of the same age wood and had lateral branches removed either in the nursery operation or just prior to testing. The carob were 48 in. long and the others 60 in. Depending on pruning practices, these lengths represent nursery-grown trees of about 5 to 7 and 6 to 9 ft height, respectively. The base of these specimens was clamped in a vise and the tip was loaded horizontally with progressively increasing weights until the point of yielding of the specimens was reached. The tip (horizontal and vertical) deflection was recorded and photographs made at each increase in loading. These data were used for verification of the computer solution.

When the results of the theoretical analysis indicated an optimum taper parameter of \(-0.60\) to \(-0.67\) for all 4 species whether field- or container-grown, additional research was done to determine how the sapling tree bends under wind loading.

A number of young, unstaked, unpruned, container-grown trees were subjected to loading of approximately 20 m.p.h. with a wind machine (2). These specimens were then loaded at single points corresponding to taper parameters of \(-0.45\), \(-0.60\) and \(-0.85\) until the deflection of the trunk at the point of loading corresponded to that of the wind-loaded specimen. All were photographed on 4 x 5 in. film. Prints were made at 8 x 10 in. enlargement and tracings of the trunk curvatures superimposed for the purpose of estimating the best fits to the wind-loaded specimens. Although 8 specimens representing 5 species were used, results of only 2 representative specimens are reported.

\textbf{Results}

\textit{Mathematical Analysis and Verification.} The terms used in the following analysis are defined by Timoshenko and Young (10). In the analysis, the sapling trunk (Fig. 1) is treated like a cantilever beam (Fig. 2a) rigidly fixed at the ground, and with a horizontal wind load \(P\) applied near the center of the crown. The question to be answered is: how does the stress in a tapered trunk compare with the stress in a non-tapered trunk? The result will depend importantly upon the moment arm \(a\), and also upon the diminishing cross-section of the trunk proceeding from the base to the tip.

For an initially straight beam, the bending equation is

\[
\frac{d\theta}{ds} = \frac{M}{EI}
\]

where \(\theta\) is the slope of the beam at a point a distance \(s\) from the origin \(O\); \(M\) is the bending moment in the beam at point \(s\); \(E\) is the modulus of elasticity of the material; and \(I\) is the moment of

\begin{table}
\centering
\begin{tabular}{|l|c|c|}
\hline
Species & Container-grown & Field-grown \\
\hline
\textit{Ceratonia siliqua} & 6-8 ft & 6-8 ft \\
\textit{Ginkgo biloba} & 6-8 ft & 5-7 ft \\
\textit{Liquidambar styraciflua} & 6-8 ft & 6-8 ft \\
\textit{Betula verrucosa} & 6-8 ft & 5-7 ft \\
\hline
\end{tabular}
\caption{Specifications of specimens used in the analysis.}
\end{table}
Table 1. Experimental specimens of sapling trees used to verify computer solution of stress distribution.

<table>
<thead>
<tr>
<th>No.</th>
<th>Species</th>
<th>( L ) (inches)</th>
<th>( L ) (in.)</th>
<th>Tip (in.)</th>
<th>( I_0 ) (in.(^4))</th>
<th>( kL )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Carob (field grown)</td>
<td>48</td>
<td>0.72</td>
<td>0.25</td>
<td>0.0132</td>
<td>-0.67</td>
</tr>
<tr>
<td>2</td>
<td>&quot; &quot; &quot; &quot; &quot;</td>
<td>&quot;</td>
<td>0.76</td>
<td>0.34</td>
<td>0.0158</td>
<td>-0.55</td>
</tr>
<tr>
<td>3</td>
<td>&quot; &quot; &quot; &quot; &quot;</td>
<td>&quot;</td>
<td>0.93</td>
<td>0.29</td>
<td>0.0360</td>
<td>-0.67</td>
</tr>
<tr>
<td>4</td>
<td>&quot; &quot; (container-grown)</td>
<td>&quot;</td>
<td>0.72</td>
<td>0.72</td>
<td>0.0032</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>&quot; &quot; &quot; &quot; &quot;</td>
<td>&quot;</td>
<td>0.67</td>
<td>0.67</td>
<td>0.0098</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>&quot; &quot; &quot; &quot; &quot;</td>
<td>&quot;</td>
<td>0.59</td>
<td>0.59</td>
<td>0.0057</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>&quot; &quot; &quot; &quot; &quot;</td>
<td>&quot;</td>
<td>0.73</td>
<td>0.40</td>
<td>0.0139</td>
<td>-0.45</td>
</tr>
<tr>
<td>8</td>
<td>&quot; &quot; &quot; &quot; &quot;</td>
<td>&quot;</td>
<td>0.61</td>
<td>0.35</td>
<td>0.0068</td>
<td>-0.45</td>
</tr>
<tr>
<td>9</td>
<td>&quot; &quot; &quot; &quot; &quot;</td>
<td>&quot;</td>
<td>0.70</td>
<td>0.40</td>
<td>0.0118</td>
<td>-0.45</td>
</tr>
<tr>
<td>10</td>
<td>&quot; &quot; (container-grown)</td>
<td>60</td>
<td>0.73</td>
<td>0.34</td>
<td>0.0139</td>
<td>-0.55</td>
</tr>
<tr>
<td>11</td>
<td>&quot; &quot; &quot; &quot; &quot;</td>
<td>&quot;</td>
<td>0.71</td>
<td>0.31</td>
<td>0.0124</td>
<td>-0.40</td>
</tr>
<tr>
<td>12</td>
<td>&quot; &quot; &quot; &quot; &quot;</td>
<td>&quot;</td>
<td>0.77</td>
<td>0.29</td>
<td>0.0169</td>
<td>-0.60</td>
</tr>
<tr>
<td>13</td>
<td>&quot; &quot; &quot; &quot; &quot;</td>
<td>&quot;</td>
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<td>0.44</td>
<td>0.0124</td>
<td>-0.40</td>
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<td>14</td>
<td>&quot; &quot; &quot; &quot; &quot;</td>
<td>&quot;</td>
<td>0.61</td>
<td>0.45</td>
<td>0.0066</td>
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<tr>
<td>15</td>
<td>&quot; &quot; &quot; &quot; &quot;</td>
<td>&quot;</td>
<td>0.64</td>
<td>0.44</td>
<td>0.0083</td>
<td>-0.30</td>
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<tr>
<td>16</td>
<td>Liquidambar (field-grown)</td>
<td>&quot;</td>
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<td>0.18</td>
<td>0.0506</td>
<td>-0.80</td>
</tr>
<tr>
<td>17</td>
<td>European birch (field grown)</td>
<td>&quot;</td>
<td>0.62</td>
<td>0.25</td>
<td>0.0092</td>
<td>-0.60</td>
</tr>
</tbody>
</table>

Inertia of the cross-section about a transverse axis, and is a function of \( s \). (See Fig. 2a.)

At a general distance \( x \) above the ground (Fig. 2b), there will exist in the trunk a longitudinal force \( T \), a transverse force \( V \), and a bending moment \( M \). In Fig. 2b, an imaginary cut has been made at \( x \); the lower portion of the trunk has been removed and has been replaced by the forces \( T, V \), and the moment \( M \), which the lower part of the trunk exerts on the upper part. If we write the equation of equilibrium of moments about the imaginary cut, we obtain

\[
M = P(a - x) \tag{2}
\]

The moment of inertia for a round cross-section, as a function of \( s \), is

\[
I = I_0(ks + 1)^4 \tag{3}
\]

where \( I_0 = \pi R^4 \) is the moment of inertia at the base,

\[
R = \text{radius of trunk at base} \quad k = \frac{(R - r)}{R L}
\]

\( r = \text{radius of trunk at point of loading} \)

\( L = \text{length of sapling to the point of loading} \)

Substituting Equation (2) and (3) into Equation (1), we have

\[
\frac{d\theta}{ds} = \frac{P(a - x)}{E I_0 (ks + 1)^4} \tag{4}
\]

Thus as one proceeds an infinitesimal distance \( ds \) up the stem, the increase in bending \((d\theta)\) will be greater with: a) a greater loading \((P)\), b) a greater distance between the tip and the point under consideration \((a - x)\), c) more flexible wood (lower modulus of elasticity, \(E\)) or d) smaller moment of inertia of the cross-section, \(I = I_0(ks + 1)^4\).

Differentiating Equation (4), we obtain

\[
\frac{d^2\theta}{ds^2} = -\frac{P}{E I_0} \frac{(ks + 1)^4}{(ks + 1)^8} \tag{5}
\]

Using Equation (4), and also the relation \(dx/ds = \cos \theta \), we get.

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Fig. 1. Schematic diagram of sapling tree used in this analysis.

Fig. 2. (a) Idealized diagram of tree trunk bending under wind load; (b) imaginary cut has been made at height \( x \), showing internal forces and bending moment at cut surface.
In order to non-dimensionalize this equation, we introduce the change of variable $\xi = s/L$, and obtain

$$\frac{d^2 \theta}{d \xi^2} = \frac{P L^2}{E I_0} \cdot \frac{\cos \theta}{(k L)^4} \left( \frac{4 (k L)}{(k L) \xi + 1} \right) \frac{d \theta}{d \xi}$$

Thus, the deflection of the beam will be a function of the dimensionless parameters $PL^2/EI_0$ and $kL$. The parameter $PL^2/EI_0$ is called the "load parameter"; for a given specimen, $L$, $E$, and $I_0$ will be constant, so the parameter will vary directly with the load $P$. The parameter $kL$ is the "taper parameter," and may vary from 0 to $-1$. The larger the magnitude of $kL$, the more severe the degree of taper. (Since $kL = \text{taper parameter} = -(R - r)/R$, it would appear superficially, that the solution does not depend upon the length $L$; however, in equation (7), it should be noted that the influence of $L$ is contained in the independent variable $\xi$, which is equal to $s/L$.)

In effect this analysis states that the bending in successive regions along a tapered stem is not uniform. If we consider the tapered stem as if divided into small pieces, each higher piece will bend a little more until the region is approached where the stem is inclined very nearly in the direction of the load.

Equation (7) is a nonlinear differential equation in $\theta$ and $\xi$. It has been solved analytically (1) for the case $kL = 0$ (i.e., for non-tapered beams), but has not been solved analytically for non-zero values of $kL$ (tapered beams). For the present analysis, a numerical solution for equation (7) was performed on a computer (6), and verified experimentally, (see tabulation of experimental test specimens in Table 1). The results are shown in Fig. 3 and 4. The trend of the experimental points follow the computed curves closely. In these figures, the physical parameter $E$ has been chosen for best fit, and the
Fig. 6. Photographs of test specimens #1 and #4 (both Ceratonia siliqua, carob). In (1), specimen is tapered with base diam = 0.72" and tip diam = 0.25" (specimen No. 1, $kL = -0.67$); in (4) specimen is without taper, diam = 0.72" (specimen #4). Both specimens are 48" long, and are loaded with $P = 4.25$ lb. (Note: load pan weighs 0.25 lb. and must be added to loads shown on photos.)

The calculation of stress in the sapling trunk now proceeds as follows. Computed values for the deflected shapes of certain ideal sapling trunks are shown in Fig. 7, for $PL^2/EI_0 = 1.8$ and $2.7$, and for $kL = 0$, $-0.30$, $-0.60$, and $-0.90$. The maximum bending stress at any point in the trunk can be calculated from the formula

$$\text{stress} = \frac{Mc}{I}$$

(8)

where $c =$ radius of trunk at point where stress is being calculated, and $M$ and $I$ are defined by Equations (2) and (3). Values for $a$ and $x$ may be obtained from Fig. 7, to be used in Equation (2).
The results of stress calculations for several typical sapling shapes are shown in Fig. 8 and 9. In Fig. 8, the assumed point of application of load for all cases is 48”; the base diam is 0.720”. In Fig. 9 the assumed point of application of load for all samples is 96” above the base; the base diam is 2.00”. (Although no specimens of this size were actually tested, data in Fig. 9 represent a young tree with light pruning of about 12 ft and one with heavy pruning of laterals of as little as 10 ft height.) In each figure, the effect of variation in taper parameter is shown. (See Table 2, for assumed dimensions of samples.) The first and fourth sample shapes used in Fig. 8 are approximately those of two of the experimental specimens (Specimens #1 and #4, Table 1). For Fig. 8, stress is calculated as a result of an assumed 4.25 lb. horizontal load, corresponding to a load parameter $\frac{P}{L^2EI_0} = 1.8$. For Fig. 9, the load parameter is also 1.8, but the horizontal load corresponding to the dimensions of these sample shapes is 62.90 lb.

An optimum taper is evident, from examination of Figs. 8 and 9. The optimum visible in both figures is for a taper with $kL$ approximately equal to $-0.60$, since this is the taper corresponding to least stress, in each case. For the sample shapes of Fig. 8, the least stress is approximately 4000 psi; for the shapes of Fig. 9 the least stress is approximately 5500 psi. Whether either of these stresses would result in breakage would depend upon the strength of the particular specimen under consideration, but these stresses are close to the strength limits of many kinds of wood. It is to be noted that a trunk with no taper would have a max stress of approximately 4800 psi (at the base) for the sample shapes of Fig. 8, and 6500 psi for those of Fig. 9. Severely tapered trunks, such as those with $kL = 0.90$, would have even higher stresses, as can be seen in Figs. 8 and 9. Several sample calculations, carried out for trees with a range base diam from 0.625”, to 2.00”, with load parameter $\frac{P}{L^2EI_0} = 1.8$, show the results given in Table 3.

Table 3 gives the ratio of the max stress in a tapered trunk, to the max stress in a non-tapered trunk, for different values of the taper parameter and for different base diam. For example, using the same values as are used in Fig. 8 (base diam = 0.72”), the max stress in a trunk with $kL = -0.60$, is 4000 psi; the max stress in a non-tapered trunk of this diam is 4800 psi; hence, the ratio is 4000/4800 = 0.83, for this base diam. In Table 3, the third line ($kL = -0.60$) shows the lowest set of values for this ratio: the most favorable value is for a base diam of 0.750”; at this combination of taper parameter, load parameter, and base diam, the max stress is only 78% of the max stress in a non-tapered trunk of the same base diam. The table further shows that taper in the trunk can reduce the max stress by about 10% if $kL = -0.30$, and by more than 20% if $kL = -0.60$, in the range of base diam considered. Higher tapers than $kL = -0.60$ have higher ratios.

### Table 2. Assumed sample shapes used in the solution of stress distribution of thin tapered beams (L = 48”).

<table>
<thead>
<tr>
<th>Base Diam (inches)</th>
<th>Tip Diam (inches)</th>
<th>kL</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.720</td>
<td>0.720</td>
<td>0</td>
</tr>
<tr>
<td>0.720</td>
<td>0.504</td>
<td>-0.30</td>
</tr>
<tr>
<td>0.720</td>
<td>0.288</td>
<td>-0.60</td>
</tr>
<tr>
<td>0.720</td>
<td>0.240</td>
<td>-0.67</td>
</tr>
<tr>
<td>0.720</td>
<td>0.072</td>
<td>-0.90</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Base Diam (inches)</th>
<th>Tip Diam (inches)</th>
<th>kL</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.000</td>
<td>2.000</td>
<td>0</td>
</tr>
<tr>
<td>2.000</td>
<td>1.400</td>
<td>-0.30</td>
</tr>
<tr>
<td>2.000</td>
<td>0.800</td>
<td>-0.60</td>
</tr>
<tr>
<td>2.000</td>
<td>0.667</td>
<td>-0.67</td>
</tr>
<tr>
<td>2.000</td>
<td>0.200</td>
<td>-0.90</td>
</tr>
</tbody>
</table>

### Table 3. Stress ratios in sapling tree trunks; load parameter $\frac{P}{L^2EI_0} = 1.8$.

| Stress ratio = Max stress in tapered trunk Max stress in non-tapered trunk Base diam |
|----------------------------------|-------------------------------|
| $kL$                             | 2.000” | 1.000” | 0.750” | 0.625” |
| 0 (no taper)                     | 1.00   | 1.00   | 1.00   | 1.00   |
| -0.30                            | 0.93   | 0.91   | 0.90   | 1.00   |
| -0.60                            | 0.84   | 0.83   | 0.78   | 0.96   |
| -0.90                            | 1.01   | 1.41   | 1.00   | 1.10   |

Fig. 8. Stress distribution in a sapling trunk as a function of taper. (For $P = 4.25\#$, base diam = 0.72”, $L = 48$”; load parameter $\frac{P}{L^2EI_0} = 1.8$.)

Fig. 9. Stress distribution in a sapling trunk as a function of taper. (For $P = 62.90\#$, base diam = 2.00”, $L = 96$”; load parameter $\frac{P}{L^2EI_0} = 1.8$.)
Fig. 10. Curvatures of Eucalyptus tessellaris under wind-loading and single point loading with loads applied at points corresponding to taper parameters, $kL = -0.45$, $-0.60$ and $-0.85$.

$-0.60$ offer little advantage, and in fact seem to be detrimental for taper parameters on the order of $-0.90$. The optimum appears to be in the neighborhood of $kL = -0.60$ for the range of parameters considered.

Wind Loaded Saplings. Comparison of wind- and single point-loading of Eucalyptus tessellaris, Firm., (Fig. 10) and Betula verrucosa, Ehrh., (Fig. 11) demonstrate that the best fit curves occur when the single-point load is applied at a point which corresponds to a taper parameter of $-0.60$. The curvature of specimens loaded at a point corresponding to a taper parameter of $-0.45$ is to the right of the wind-loaded specimen below the point of loading and crosses sharply to the left above that point, i.e. most of the curvature is between the base and $0.45$ the total height of the specimen.

The curvature of specimens loaded at a point corresponding to a taper parameter of $-0.85$ is to the left of the wind-loaded specimen with sharp bending occurring in the upper portion of the sapling.

Discussion

In the verification of the computer analysis, the specimen trunks used had all branches and leaves removed. Although the loading was applied at the tip of the specimen it should be emphasized that this point corresponds to the assumed point of wind loading in a growing tree. The case where taper parameter equals $0.00$ is often met in container-grown trees. The case where taper parameter equals $-0.90$ would only occur if all limbs and most of the leaves on the lower portion of a tree with uniform taper were removed. For example, a 10.0-ft tree with uniform taper and with all laterals removed to such a height (about 8 1/2 ft) that the loading was at about 9.0 ft, would have a taper parameter of $-0.90$.

From a mechanical point of view the optimum structural member is the one in which there is the most uniform distribution of stress. From a biological point of view stress should be distributed uniformly in the region of the trunk where wood development is most advanced and should decrease rapidly where wood development (lignification, cell wall thickening etc.) is not complete. This condition appears to be best met in the case where the taper parameter is approximately $-0.60$ (Figs. 8 and 9).

The bending of the trunk in the sapling tree is strongly influenced by the taper (Figs. 3-7). The fact that a tapered trunk deflects further than a non-tapered one, thus producing a smaller moment arm, $a$ (see Fig. 2), is an important effect which results in a lower max stress in a tapered trunk.

The validity of this mathematical solution is demonstrated by the close agreement of the computer solution (Fig. 5, solid lines) to the experimental data (Fig. 5, experimental points and

Evaluation and Measurement of Characteristics Affecting Fresh Market Blueberry Demand

G. A. Mathia, and R. A. Schrimper

North Carolina State University, Raleigh

Abstract. Fresh market demand relationships were estimated for several major markets for North Carolina, New Jersey, and Michigan fresh blueberries. Demand relationships were estimated from daily price and unload statistics for the 1965-1971 seasons by regression analysis using daily price as the dependent variable. Zero-one variables were used to account for the seasonal and within season effects.

The relationships accounted for 50 to 70% of the variation in daily prices during the 7-year period. The remaining unexplained variation resulted in standard errors of estimate of around 15 cents per flat for most markets. Large standard errors associated with the quantity coefficients appeared to be related in part to the relatively large and discrete intervals used in price reporting by the Market News Service. The quantity coefficients were relatively small. These indicated rather elastic demands for fresh blueberries in the major markets.

A large share of blueberries for the fresh market is produced in New Jersey, North Carolina, and Michigan. Also, most fresh market blueberries shipped from the 3 major supply areas are marketed in a few large metropolitan areas of the Northeast and Midwest. The 5 market areas listed in Table 1 accounted for nearly 60% of total unloads of fresh blueberries in recent years. Significant price differences occurred in the selected markets.

Literature Cited